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The Riemann Surface of the Logarithm Constructed in a Geometrical Framework

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Abstract

The logarithmic Riemann surface Σ_{\log} is a classical holomorphic 1-manifold. It lives into \mathbb{R}^4 and induces a covering space of $\mathbb{C} \setminus \{0\}$ defined by $\exp_{\mathbb{C}}$.

This paper suggests a geometric construction of it, derived as the limit of a sequence of *vector fields extending* $\exp_{\mathbb{C}}$ suitably to *embeddings of* \mathbb{C} *into* \mathbb{R}^3 , which turn to be *helicoid* surfaces living into $\mathbb{C} \times \mathbb{R}$. In the limit we obtain a bijective complex exponential on the covering space in question, and thus a well-defined complex logarithm. In addition, the *helicoids* are *diffeomorphic (not bi-holomorphic) copies* of Σ_{\log} as C^∞ -realizations living into \mathbb{R}^3 , without obstruction.

Our approach is purely *geometrical* and *does not employ any tools provided by the complex structure*, thus *holomorphy is no longer necessary to obtain constructively this Riemann surface* Σ_{\log} . Moreover, the differential geometric framework we adopt affords explicit generalization on submanifolds of $\mathbb{C}^m \times \mathbb{R}^m$ and certain corollaries are derived.

Keywords: Riemann Surface of the Logarithm, surfaces and submanifolds of Euclidean space.

MSC Subject Classification 2000: 33B30, 33B10, 37E30, 53A05, 53A07.

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*"You can not claim you are well aware of a theorem
unless you have more than one proof for it"*

M. Atiyah's dictum

Introduction.

Riemann surfaces have been, at least conceptually, first introduced by Riemann in his celebrated 1851 PhD dissertation. He considered a surface spread over \mathbb{C} with several sheets lying over it, in such a way that a complex multi-valued function with branches becomes a true well defined function, mapping each branch to a sheet. His posterity gave a rigorous definition of the (so called) *Concrete Riemann surfaces* as covering spaces of \mathbb{C} (for an enlightening review see [18]).

Klein was the first to put away the covering space approach and adopted a differential geometric one: he studied complex functions living on a curved surface of some ambient Euclidean space. In fact, it was the first time the *Atlas of holomorphic structure* was introduced. At those times, "Riemann surface" meant compact 2-manifold with an arc-length element ds^2 and bi-holomorphic transition functions in the atlas. Several years later, adding Cantor's 2nd countability and Hausdorff's axioms (not before the 1920's), Radó reaches via triangulations the *Abstract Riemann surfaces'* definition: a *Hausdorff 2nd-countable topological surface with complex structure*.

The equivalence of *Concrete* and *Abstract* Riemann surfaces follows from a theorem of Behnke & Stein (1947–49) [2], improved by Gunning and Narasimhan (1967) [8].

In modern times, the research interest has been transferred to the study of their *Moduli* (*ℰ Teichmüller*) *spaces*, being in effect sets of equivalence classes of distinct complex structures, modulo the action of orientation preserving diffeomorphism of the surface with certain topological structures [13], as well as to their vasty applications in *theoretical physics*, specifically *M-theory unifying the several String theories* (see e.g. [12]).

In this paper we follow a semi-classical approach, standing on the line between the *abstract* and the *concrete*. Our principal result is that *we recover in a pure differential-geometric fashion results classically obtained via the holomorphic*

structure itself, bypassing analytic continuation.

Technically, the employed apparatus consists of a sequence of vector fields $\{\text{Exp}_n\}_{n \in \mathbb{N}}$ defined on \mathbb{C} and valued in (the tangent bundle of) \mathbb{R}^3 . Each field $\text{Exp}_n : \mathbb{C} \longrightarrow \mathbb{C} \setminus \{0\} \times \mathbb{R}$ (*Def. 1.3*) geometrically is in fact a smooth parametric surface of \mathbb{R}^3 called (exponential) helicoid (*Def. 1.1*). These Exp_n 's extend (due to the extra component) the ordinary $\exp_{\mathbb{C}}$ in such a way that they become C^∞ -diffeomorphisms onto their images (& embeddings of \mathbb{C} into $\mathbb{C} \times \mathbb{R}$) removing $2\pi i$ periodicity.

The Exp_n -maps by construction converge to the covering space of the punctured plane $\exp_{\mathbb{C}} : \mathbb{C} \longrightarrow \mathbb{C} \setminus \{0\}$. Bijectivity is preserved in the limit allowing a well defined limit logarithm $\log_{\mathbb{C}}$ to be introduced on the direct limit surface $\varinjlim \text{Exp}_n(\mathbb{C}) \cong \exp_{\mathbb{C}}(\mathbb{C})$ (in 2.7) which essentially is its *Riemann surface* Σ_{\log} immersed into the punctured plane. Even though all the countably many sheets are "compressed" to a single sheet, we still have the correspondence of the $\log_{\mathbb{C}}$ -branches and the sheets of the covering space via the convergence mechanism.

Notwithstanding, we show that each Exp_n is C^∞ -diffeomorphic to Σ_{\log} , when the latter is considered as equipped with the smooth sub-atlas of the holomorphic one coming from the ambient \mathbb{R}^4 -space. The fact that the helicoids can only be diffeomorphic copies of Σ_{\log} and not bi-holomorphic is imposed by topological obstructions due to the (general) non-imbeddability of Riemann surfaces into \mathbb{R}^3 .

The last *Section* is devoted to high-dimensional generalizations on multi-helicoid submanifolds of Euclidean $3m$ -space.

In view of the above, a natural question perhaps arises to the reader: what kind of outstanding property do the helicoids enjoy and what is so special about the Exp vector fields? The answer is that they are an *ad hoc* choice in order to provide the desired properties via convergence. There may very well exist even more privileged surfaces of \mathbb{R}^3 , but the smooth realizations of the holomorphic Σ_{\log} into 3-space are *dimension-wise* clearly *optimal*.

1. Preliminaries.

We collect a few preparatory results which will be needed in the main part of the paper, noted here for the reader's convenience.

1.1 Definition. (Helicoid) The smooth (parametric) surface of the (*exponential*) *helicoid* (see e.g. [17], [19]), is the map $X : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ given by

$$(u, v) \longmapsto (e^u \cos v, e^u \sin v, av), \quad a > 0$$

where a is a parameter. We consider this as the (global) coordinate system of a 2-manifold imbedded in \mathbb{R}^3 . In complex coordinates we may write

$$X : \mathbb{C} \longrightarrow \mathbb{C} \times \mathbb{R} \quad :$$

$$u + iv \longmapsto (\exp_{\mathbb{C}}(u + iv), av) \equiv (\exp_{\mathbb{C}}(z), a\operatorname{Im}(z))$$

where $z \equiv u + iv$. The complex functions $Re, Im, \exp_{\mathbb{C}} : \mathbb{C} \longrightarrow \mathbb{C}$ denote the "real part", the "complex part" and the "complex exponential" of \mathbb{C} respectively.

The terminologies of imbeddings and immersions we follow are the standard ones, referring e.g. to [7], [11].

We employ the convention that "*imbedding*" stands for topological imbedding and "*embedding*" for geometrical embedding in the sense of smooth manifolds.

In this paper *smoothness* means C^∞ - *smoothness* in the usual geometric sense and diffeomorphism stands for C^∞ -diffeomorphism.

Furthermore, as usual by the term *smooth n -manifold M* (and in particular, surface) we mean a C^∞ -smooth connected, paracompact Hausdorff manifold, of real n dimensions $\dim_{\mathbb{R}}(M) = n$ (2, in the 2nd case).

1.2 Remark. Let now

$$\exp_{\mathbb{C}} : \mathbb{C} \longrightarrow \mathbb{C} \setminus \{0\} \quad : \quad z \longmapsto \exp_{\mathbb{C}}(z) \equiv e^z := \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

be the ordinary complex exponential function (*the notations $\exp_{\mathbb{C}}(z)$ and e^z will be occasionally exchanged without comments*). From the differential-geometric viewpoint, this map may be considered as a vector field, tangent on the 2-dimensional (flat) manifold $\mathbb{C} \cong \mathbb{R}^2$, i.e.

$$\mathbb{C} \equiv \mathbb{R}^2 \ni ue_1 + ve_2 \longmapsto (e^u \cos v)e_1 + (e^u \sin v)e_2.$$

This owes to the fact that if $\xi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a vector field of \mathbb{R}^n (in the *vector calculus* sense), where $x \longmapsto \xi(x)$, then the obvious identification $\xi(x) \equiv (x, \xi(x))$ is the requested one in order to consider ξ valued in the tangent bundle $T\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n \cong \mathbb{R}^{2n}$ (vector field in the geometric sense) by global triviality of the vector (tangent in our case) bundle ([11], [7], [20], etc).

The current task is to define a certain extension of the exponential $\exp_{\mathbb{C}}$ from \mathbb{C} to $\mathbb{C} \times \mathbb{R}$ that removes the $2\pi i$ -periodicity and become injective due to extra real component.

These remarks introduce the idea of the following definition:

1.3 Definition. (The exponential field) The exponential vector field on \mathbb{C} is the map

$$\text{Exp}_a : \mathbb{C} \longrightarrow \mathbb{C} \times \mathbb{R}$$

given by:

$$u + iv \longmapsto \text{Exp}_a(u + iv) := (\exp_{\mathbb{C}}(u + iv), av)$$

where $a > 0$ is a parameter to be fixed as will in the sequel. Operationally the map can given as

$$\text{Exp}_a = (\exp_{\mathbb{C}}, a \text{ Im})$$

The following technical fact shows that this vector-wise extension of the $\exp_{\mathbb{C}}$ function on \mathbb{C} is the requested one that provides injectivity of the map, now considered as a vector field.

1.4 Proposition. (Structural Properties of Exp)

The map $\text{Exp}_a : \mathbb{C} \longrightarrow \mathbb{C} \times \mathbb{R}$ given by 1.3 has the following properties:

- a) *It is a vector field defined on the (trivially) embedded submanifold $\mathbb{C} \equiv \mathbb{C} \times \{0\} \hookrightarrow \mathbb{C} \times \mathbb{R} \cong \mathbb{R}^3$ and valued into the (tangent bundle of the) ambient space \mathbb{R}^3 .*
- b) *It is a diffeomorphism onto its image, thus an embedding of the plane into $\mathbb{C} \times \mathbb{R}$ and, in particular, invertible.*

As our context suggests, we consider submanifolds as subsets of \mathbb{R}^n , where the inclusion map is an imbedding and the Atlas that determines differential structure and topology is the induced from \mathbb{R}^n .

Proof. a) We recall that a vector field defined on a *submanifold* M of \mathbb{R}^n is not necessarily tangent to the manifold in the usual differential geometric sense, but the splitting of tangent spaces

$$\mathbb{R}^n \cong T_p \mathbb{R}^n \cong T_p M \bigoplus T_p^\perp M, \quad p \in \mathbb{R}^n$$

implies that it can be normal, or generally valued in $T\mathbb{R}^n \cong \mathbb{R}^{2n}$. The very definition of Exp_a implies that it maps a point z of \mathbb{C} to a vector $\text{Exp}_a(z)$ of $\mathbb{C} \times \mathbb{R}$

$$\mathbb{C} \ni z \longmapsto \text{Exp}_a(z) \in \mathbb{C} \times \mathbb{R}$$

and *Remark* 1.2 implies that the map is a vector field of \mathbb{C} valued in the tangent bundle of $\mathbb{C} \times \mathbb{R}$.

b) If $\text{Exp}_a(u_1 + iv_1, av_1) = \text{Exp}_a(u_2 + iv_2, av_2)$ for $u_1, u_2, v_1, v_2 \in \mathbb{R}$, then this amounts to

$$(\exp_{\mathbb{C}}(u_1 + iv_1), av_1) = (\exp_{\mathbb{C}}(u_2 + iv_2), av_2)$$

that is

$$\exp_{\mathbb{R}}(u_1) \exp_{\mathbb{C}}(iv_1) = \exp_{\mathbb{R}}(u_2) \exp_{\mathbb{C}}(iv_2), \quad v_1 = v_2$$

and consequently $u_1 = u_2$ by the monotonicity of the real exponential. This shows the injectivity of Exp , thus it is bijective onto its image $\text{Exp}_a(\mathbb{C})$. C^∞ -differentiability of this map goes without saying, since it has smooth component functions in $\mathbb{C} \times \mathbb{R}$. Smoothness of the inverse map is also obvious (for the explicit expression of the inverse map see *Lemma* 2.6 of the oncoming *Section 2*). Consequently, the map is a smooth (geometric) embedding of \mathbb{C} into $\mathbb{C} \times \mathbb{R}$. \square

Consider now the standard Euclidean (Riemannian) metric δ of \mathbb{R}^3 with components the Kronecker deltas' δ_{ab} (where $\delta_{aa} = 1$ and 0 otherwise, $1 \leq a \leq 3$) (for the standard concepts of *Riemannian geometry* employed in this paper, we refer to [5], [7], [11], [17]). Then, the (trivially) embedded surface \mathbb{C} gets a natural induced *flat* Riemannian metric, the standard inner product $\langle \cdot, \cdot \rangle \equiv \delta$ of \mathbb{R}^3 by restriction of δ on \mathbb{C} , which can be seen as the pull-back metric via the inclusion map:

$$(\mathbb{C}, \delta|_{\mathbb{C}}) \equiv (\mathbb{C}, i^*(\delta))$$

$$i : \mathbb{C} \hookrightarrow (\mathbb{C} \times \mathbb{R}, \delta)$$

In view of theses remarks, we obtain the next result giving some geometric properties of the extended object Exp_a as a vector field on the $(\mathbb{C}, i^*(\delta))$.

Notation. $\text{angl}(\xi, \eta)(p)$ denotes the angle of 2 vector fields ξ, η measured at the point p of the surface (manifold), with respect to its Riemannian metric.

1.5 Proposition. (*Geometric Properties of Exp*)

a) *The vector field $\text{Exp}_a : \mathbb{C} \longrightarrow \mathbb{C} \times \mathbb{R}$ geometrically is a smooth exponential helicoid surface of the form 1.1 and the following diagram commutes:*

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\text{Exp}_a} & \text{Exp}_a(\mathbb{C}) \\ \downarrow \text{exp}_{\mathbb{C}} & & \downarrow i \\ \mathbb{C} \setminus \{0\} & \xleftarrow{\pi_{\mathbb{C} \setminus \{0\}}} & \mathbb{C} \setminus \{0\} \times \mathbb{R} \end{array}$$

that is, Exp_a commutes with the projection composed with the inclusion of the surface $i : \text{Exp}_a(\mathbb{C}) \hookrightarrow \mathbb{C} \setminus \{0\} \times \mathbb{R}$ and the projection $\pi_{\mathbb{C} \setminus \{0\}} : \mathbb{C} \setminus \{0\} \times \mathbb{R} \longrightarrow \mathbb{C} \setminus \{0\}$

$$(\pi_{\mathbb{C} \setminus \{0\}} \circ i) \circ \text{Exp}_a = \text{exp}_{\mathbb{C}}$$

b) *If the tangent plane $T_{u+iv}\mathbb{C} \cong \mathbb{C} \equiv \mathbb{R}^2$ is spanned by $e_1 = (1, 0)$ and $e_2 = (0, 1)$ and*

$$\xi_{AB} \equiv Ae_1 + Be_2$$

is a general 2-parameter family of tangent vectors, $A, B \in \mathbb{R}$, $|A| + |B| \neq 0$, then

$$\cos[(\Phi_{AB})(u, v)] = \frac{e^u [A \cos v + B \sin v]}{\sqrt{e^{2u} + a^2 v^2} \sqrt{A^2 + B^2}}$$

where $\Phi_{AB}(u, v)$ is the angle of Exp with ξ_{AB} at $u + iv$ in \mathbb{R}^3 :

$$(\Phi_{AB})(u, v) \equiv \text{angl}(\text{Exp}_a, \xi_{AB})(u + iv)$$

1.6 Remark. The calculation of the aforementioned (non-constant) angle shows that the Exp_a -vector field is neither tangent nor normal to \mathbb{C} .

Proof. a) The first claim goes without saying, just by comparing the very definitions 1.1 and 1.3. Thus, the map Exp_a is a vector field of \mathbb{R}^3 defined on \mathbb{C} and simultaneously its image constitutes a smoothly embedded surface of \mathbb{R}^3 , an exponential helicoid diffeomorphic to \mathbb{C} .

Commutativity is a simple consequence of the form of the Exp_a -field:

$$(\pi_{\mathbb{C} \setminus \{0\}} \circ i) \circ \text{Exp}_a(z) = (\pi_{\mathbb{C} \setminus \{0\}} \circ (\exp_{\mathbb{C}}, a \text{ Im}))(z) = \exp_{\mathbb{C}}(z)$$

b) We recall the familiar Euclidean formula giving the angle of 2 vectors in $(\mathbb{R}^3, \delta) = (\mathbb{R}^3, \langle \cdot, \cdot \rangle)$

$$\cos [\text{angl}(\text{Exp}_a, \xi_{AB})(u, v)] = \left(\frac{\langle \text{Exp}_a, \xi_{AB} \rangle}{\|\text{Exp}_a\| \|\xi_{AB}\|} \right)(u + iv)$$

The 2-parameter vector (describing a general tangent vector) is given due to the inclusion $\mathbb{C} \times \{0\} \hookrightarrow \mathbb{C} \times \mathbb{R}$ by the formula

$$\xi_{AB} = Ae_1 + Be_2 + 0e_3 = A(1, 0, 0) + B(0, 1, 0) + (0, 0, 0) = (A, B, 0)$$

and provided that $\text{Exp}_a = (e^u \cos v, e^u \sin v, av)$ we have

$$\cos[(\Phi_{AB})(u, v)] = \frac{A e^u \cos v + B e^u \sin v}{\sqrt{e^{2u} + a^2 v^2} \sqrt{A^2 + B^2}}$$

which gives the requested formula and this completes the proof. \square

2. The Logarithmic Riemann surface.

(Part I) The Convergence Constructions.

The *infinite-sheeted Riemann surface* Σ_{\log} of the logarithm $\log_{\mathbb{C}}$ (see [1], [9], [22]), considered as a holomorphic manifold of (complex) dimension $\dim_{\mathbb{C}}(\Sigma_{\log}) = 1$, lives into $\mathbb{C}^2 \cong \mathbb{R}^4$ and can be given as the graph of the function $e^z = w$

$$\Sigma_{\log} := \{z \in \mathbb{C} / e^z = w\} \subseteq \mathbb{R}^4$$

or by the parametric equations

$$\mathbb{C} \ni z \longmapsto (z, \exp_{\mathbb{C}}(z)) \in \mathbb{C} \times \mathbb{C} \cong \mathbb{C}^2$$

$\Sigma_{log}^{\mathcal{O}}$ denotes the holomorphic manifold $(\Sigma_{log}, \mathcal{A}_{\Sigma_{log}}^{\mathcal{O}}) \hookrightarrow (\mathbb{R}^4, id)$ with the induced analytic atlas coming from \mathbb{R}^4 . Classically, it is obtained by analytic continuation, extending holomorphically $\exp_{\mathbb{C}}$ to "small" discs along a disk $\mathbb{D}(0, 1)$ of \mathbb{C} . Since the last extension that overlaps with the first does not coincide with it, we take a copy of \mathbb{C} and proceed continuation to a next sheet lying over the initial one. Continuing ad infinitum, we obtain a covering space of $\mathbb{C} \setminus \{0\}$ bi-holomorphic to this surface.

Projecting to the second factor, we obtain the covering space map on the punctured plane $\mathbb{C} \setminus \{0\}$ (equivalence of *concrete* & *abstract* approach, [2], [8]):

$$\pi : \Sigma_{log}^{\mathcal{O}} \subseteq \mathbb{C}^2 \longrightarrow \mathbb{C} \setminus \{0\} : (z, \exp_{\mathbb{C}}(z)) \longmapsto \exp_{\mathbb{C}}(z)$$

In general there is no way to holomorphically imbed this 2-manifold into \mathbb{R}^3 , unless we allow self-intersections, which is no longer a homeomorphism onto its image but may still be an immersion.

Our construction allows, as we shall see, to introduce a well-defined complex logarithm. Note that *all this apparatus manages to bypass the holomorphic structure of this complex manifold and no analytic continuation is anywhere used*.

In this section we give the *geometric* construction of Σ_{log} . As stated in the *Introduction*, it is obtained as a special covering manifold of $\mathbb{C} \setminus \{0\}$ in the limit of a sequence obtained by the exponential Exp_a images of \mathbb{C} .

Hence, we recall the map $\text{Exp}_a : \mathbb{C} \longrightarrow \mathbb{C} \times \mathbb{R}$ given by 1.3 and substitute the positive parameter by the sequence $a_n = 1/n$, $n \in \mathbb{N}$. Thus, we obtain a sequence of vector fields (and surfaces of \mathbb{R}^3 as well) $\{\text{Exp}_n\}_{n \in \mathbb{N}}$, where

$$\text{Exp}_n : \mathbb{C} \longrightarrow \mathbb{C} \times \mathbb{R} : z \longmapsto \left(\exp_{\mathbb{C}}(z), \frac{1}{n} \text{Im}(z) \right), \quad n \in \mathbb{N}$$

The following result is a well-known fact and can be found in any elementary textbook, e.g. [15]. For the definition of *covering manifolds* we refer to [7], [21].

2.1 Lemma. *The pair $(\exp_{\mathbb{C}}, \mathbb{C})$, constitutes a covering manifold of $\mathbb{C} \setminus \{0\}$ with countably infinite covering sheets.*

We are now in position to prove the main result of this section. The sequence of maps $\{\text{Exp}_n : \mathbb{C} \longrightarrow \mathbb{C} \times \mathbb{R}\}_{n \in \mathbb{N}}$ is a diffeomorphism onto its image, by *Proposition* 1.4, for $a = 1/n$, $n \in \mathbb{N}$ and consequently a bijection.

We shall presently see that this property is preserved in the limit $n \rightarrow \infty$. The limit surface is the covering space produced by $\exp_{\mathbb{C}} : \mathbb{C} \rightarrow \mathbb{C} \setminus \{0\}$ (Lemma 2.1), and introducing the natural inclusion $\mathbb{C} \setminus \{0\} \hookrightarrow \mathbb{C} \setminus \{0\} \times \mathbb{R}$ we may identify $\lim_{n \rightarrow \infty} (\text{Exp}_n)$ on \mathbb{C} with the usual complex exponential, $\exp_{\mathbb{C}}$. Thus, we construct an exponential injective on \mathbb{C} .

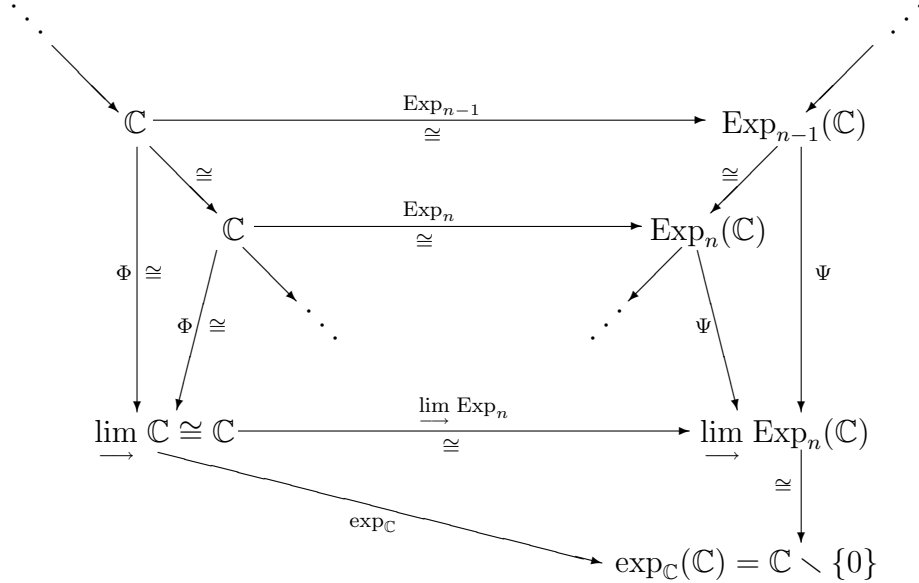
This situation resembles and simultaneously extends the analogous case in elementary complex analysis (see e.g. [1]), where $\exp_{\mathbb{C}}$ is injective and invertible on the strips of $2\pi i$ -width only.

2.2 Theorem. (*Convergence to Injective Complex Exponential*)

The sequence of diffeomorphisms $\{\text{Exp}_n : \mathbb{C} \xrightarrow{\cong} (\text{Exp}_n)(\mathbb{C}) \subseteq \mathbb{C} \setminus \{0\} \times \mathbb{R}\}_{n \in \mathbb{N}}$ converges (point-wise) to a diffeomorphism $\lim_{n \rightarrow \infty} \text{Exp}_n$ on the covering surface of $\mathbb{C} \setminus \{0\}$ produced by $\exp_{\mathbb{C}} : \mathbb{C} \rightarrow \mathbb{C} \setminus \{0\}$:

$$\text{Exp}_n \xrightarrow[pw]{n \rightarrow \infty} \exp_{\mathbb{C}} \times \{0\} \equiv \exp_{\mathbb{C}}$$

and the lower triangle of the following diagram defines (in the limit) a bijective complex exponential $\exp_{\mathbb{C}}$.



2.3 Remarks. Before giving the proof of this result which essentially contains the requested construction, some explanatory comments are necessary. First of all, the direct (inductive) limits " \varinjlim " (for their definition in a general *Category* we refer to the classical textbook [14]) are justified as follows:

The constant sequence of smooth manifolds

$$\mathbb{C} \xrightarrow[\cong]{id} \mathbb{C} \xrightarrow[\cong]{id} \mathbb{C} \longrightarrow \dots$$

defines in a completely trivial way a direct limit set which is \mathbb{C} , since $id \circ id = id$. In addition,

$$\lim_{\longrightarrow} \mathbb{C} = \lim_{n \rightarrow \infty} \mathbb{C} = \mathbb{C}$$

that is, the direct limit coincides with the usual sequence convergence. Similarly, we have the sequence

$$\text{Exp}_n(\mathbb{C}) \xrightarrow[\cong]{\Theta_{n,n+1}} \text{Exp}_{n+1}(\mathbb{C}) \xrightarrow[\cong]{\Theta_{n+1,n+2}} \text{Exp}_{n+2}(\mathbb{C}) \longrightarrow \dots$$

where the diffeomorphisms $\Theta_{n,m}$, $n, m \in \mathbb{N}$ are defined as

$$\Theta_{n,m} : \text{Exp}_n(\mathbb{C}) \longrightarrow \text{Exp}_m(\mathbb{C})$$

by the formula

$$\begin{aligned} (\exp_{\mathbb{C}}(z), \frac{1}{n}Im(z)) &\longmapsto \Theta_{n,m}(\exp_{\mathbb{C}}(z), \frac{1}{n}Im(z)) \\ &:= (\exp_{\mathbb{C}}(z), [\frac{n}{m}] \frac{1}{n}Im(z)) \end{aligned}$$

The direct limit properties can be easily verified, since by construction of the Θ -maps we get

$$\Theta_{n,m} \circ \Theta_{m,k} = \Theta_{n,k}$$

as well as

$$\Theta_{n,n} = id_{\text{Exp}_n(\mathbb{C})}$$

This implies that

$$\lim_{\longrightarrow} \text{Exp}_n(\mathbb{C}) = \lim_{n \rightarrow \infty} \text{Exp}_n(\mathbb{C}) = \exp_{\mathbb{C}}(\mathbb{C}) \times \{0\} \cong \mathbb{C} \setminus \{0\}$$

the Φ -maps pictured in the diagram are the identity (up to diffeomorphism) and the Ψ -maps are the projections on $\mathbb{C} \setminus \{0\}$ composed with the inclusions of $\text{Exp}_n(\mathbb{C})$ into $\mathbb{C} \times \mathbb{R}$:

$$\Psi := \pi_{\mathbb{C} \setminus \{0\}} \circ i$$

Proof. *Proposition 1.4* implies that Exp_n is injective, $n \in \mathbb{N}$. Consequently, we have

$$\text{Exp}_n(z) = \text{Exp}_n(w) \implies z = w$$

Our task is to prove the following condition:

$$\exp_{\mathbb{C}}(z) = \exp_{\mathbb{C}}(w) \implies z = w$$

when z, w live into \mathbb{C} and by *a)* of *Proposition 1.5* we obtain

$$\text{Exp}_n \xrightarrow{n \rightarrow \infty} \exp_{\mathbb{C}} \times \{0\} \equiv \exp_{\mathbb{C}}$$

having used that $(1/n) \xrightarrow{n \rightarrow \infty} 0$.

The well-known triangle inequality gives

$$\begin{aligned} \|\text{Exp}_n(z) - \text{Exp}_n(w)\| &\leq \|\text{Exp}_n(z) - (\exp_{\mathbb{C}}(z), 0)\| + \\ &+ \|\text{Exp}_n(w) - (\exp_{\mathbb{C}}(w), 0)\| + |\exp_{\mathbb{C}}(w) - \exp_{\mathbb{C}}(z)| \end{aligned}$$

When $n \rightarrow \infty$, we have

$$\|\text{Exp}_n(z) - (\exp_{\mathbb{C}}(z), 0)\| \xrightarrow{n \rightarrow \infty} 0$$

as well as

$$\|\text{Exp}_n(w) - (\exp_{\mathbb{C}}(w), 0)\| \xrightarrow{n \rightarrow \infty} 0$$

$$\|\text{Exp}_n(z) - \text{Exp}_n(w)\| \leq |\exp_{\mathbb{C}}(z) - \exp_{\mathbb{C}}(w)| \xrightarrow{n \rightarrow \infty} 0$$

which means that in the limit $z = w$ by the injectivity assured by 1.4. This completes the proof. \square

2.4 Corollary. *The convergence is uniform on the bounded strips of \mathbb{C} with bounded imaginary part $B \equiv \{z \in \mathbb{C} / \text{Im}(z) < M, M > 0\}$, that is, for finite spirals of the Exp_n -hellicoids:*

$$\text{Exp}_n|_B \xrightarrow[n \rightarrow \infty]{U} \exp_{\mathbb{C}} \times \{0\}|_B \equiv \exp_{\mathbb{C}}|_B$$

for any $M > 0$.

2.5 Remark. The bounded strips of the form of B corresponds in the limit to a finite-sheeted covering space of $\mathbb{C} \setminus \{0\}$:

$$\exp_{\mathbb{C}}|_B : B \longrightarrow \mathbb{C} \setminus \{0\}$$

produced by the complex exponential.

Proof. In order to see when the convergence of Exp_n to $\exp_{\mathbb{C}}$ is uniform, let $\varepsilon > 0$. Then, *Th.* 2.2 implies (using *a*) of *Prop.* 1.5)

$$\|\text{Exp}_n(z) - (\exp_{\mathbb{C}}(z), 0)\| = \|(\exp_{\mathbb{C}}(z), \frac{1}{n}\text{Im}(z)) - (\exp_{\mathbb{C}}(z), 0)\| = \frac{1}{n}|\text{Im}(z)|$$

Imposing $|\text{Im}(z)| < M$ bounded, we obtain uniform convergence as claimed. \square

The injectivity of the Exp provided by *Proposition* 1.4 gives the ability to introduce the inverse map (which we shall denote by Log) as a well-defined generalization of the complex logarithm:

$$\text{Log}_n : \text{Exp}_n(\mathbb{C}) \subseteq \mathbb{C} \setminus \{0\} \times \mathbb{R} \longrightarrow \mathbb{C}$$

where

$$\text{Log}_n := \text{Exp}_n^{-1}, \quad n \in \mathbb{N}$$

2.6 Lemma. *The inverse Log_n maps are given by*

$$(K, L) \longmapsto \text{Log}_n(K, L) = \ln(|K|) + inL$$

(\ln denotes the real Napierian logarithm).

Proof. The proof is a calculation. Set

$$\exp_{\mathbb{C}}(z) \equiv K, \quad \frac{1}{n}\text{Im}(z) \equiv L$$

Then,

$$z = \log_{\mathbb{C}}(K) = \ln(|K|) + i \arg(K)$$

Comparing the last relation with the formula $\log_{\mathbb{C}}(K) = \ln(|\text{Re}(z)|) + inL$ we conclude to the requested relation. \square

We are now in position to introduce a well-defined complex logarithm on the limit covering space produced by the exponential.

Notational convention: In the sequel we adopt the following convention: $\Sigma_{log}^{C^\infty}$ denotes Σ_{log} equipped with the induced smooth sub-atlas of the holomorphic one coming from \mathbb{R}^4 :

$$\Sigma_{log}^{C^\infty} \equiv (\Sigma_{log}, \mathcal{A}_{\Sigma_{log}}^{C^\infty}) \hookrightarrow (\Sigma_{log}, \mathcal{A}_{\Sigma_{log}}^{\mathcal{O}}) \subseteq (\mathbb{R}^4, Id)$$

2.7 Corollary. (*Convergence to a well-defined $\log_{\mathbb{C}}$ on its Riemann Surface*)

The sequence of maps $(\text{Log}_n)_{n \in \mathbb{N}}$ of 2.6 converges point-wise to a well-defined complex logarithm and uniformly on the subsets of $\text{Exp}_n(\mathbb{C}) \subseteq \mathbb{C} \setminus \{0\} \times \mathbb{R}$ of the form $\text{Exp}_n(\mathbb{C}) \cap \mathbb{C} \setminus \{0\} \times (a, b)$:

$$\text{Log}_n \xrightarrow{n \rightarrow \infty} (\exp_{\mathbb{C}} \times \{0\})^{-1} \equiv \log_{\mathbb{C}}|_{\exp_{\mathbb{C}}(\mathbb{C})}$$

and the limit surface is diffeomorphic to the (immersed into $\mathbb{C} \setminus \{0\}$) Riemann Surface of the logarithm:

$$\pi_{\mathbb{C}}(\Sigma_{log}^{C^\infty}) \cong \varinjlim (\text{Exp}_n(\mathbb{C})) = \exp_{\mathbb{C}}(\mathbb{C}) \equiv \mathbb{C} \setminus \{0\}$$

Proof. Every set contained into $\text{Exp}_n(\mathbb{C})$ (for any $n \in \mathbb{N}$) of this form with bounded height (that is, \mathbb{R} -component) is contained into a sub-helicoid $\text{Exp}_n(B) \subseteq \text{Exp}_n(\mathbb{C})$ of finite height, thus under its inversed image via Exp_n into a bounded strip B of \mathbb{C} , for some $M > 0$.

The diffeomorphism comes from the previous *Th.* 2.2 and the construction of injective complex exponential on $\exp_{\mathbb{C}}(\mathbb{C})$, as well as the remarks in the beginning of the present *Section* concerning the covering space approach. We note that the covering space map coincides with the immersion map into $\mathbb{C} \setminus \{0\}$. \square

(Part II) The Realization Theorem.

We have not so far shown the explicit relation between the surfaces $\text{Exp}_a(\mathbb{C})$ and Σ_{log} .

We recall that we manage to construct an injective $\exp_{\mathbb{C}}$ and thus a well-defined $\log_{\mathbb{C}}$ on $\text{Exp}_a(\mathbb{C}) \subseteq \mathbb{R}^3$ via convergence of helicoids. Taking into consideration the initial "defining characterization" of B. Riemann himself on the surfaces of multi-valued holomorphic functions (as the one of $\log_{\mathbb{C}}$), "surface onto which the function considered becomes single-valued" (see also [18]), we conclude that $\text{Exp}_a(\mathbb{C})$ and Σ_{\log} must necessarily be different representations of the "same" object, but the former living into \mathbb{R}^3 instead of the latter which lives into \mathbb{R}^4 .

This indeed turns to be the case, where "same" is expounded due to the dimensional reduction to a realizable level (of 3-dimensions) as follows:

2.8 Theorem. (The Realization of Σ_{\log} into \mathbb{R}^3 .) *The following diffeomorphism holds true*

$$\Xi : \text{Exp}_a(\mathbb{C}) \xrightarrow{\cong} \Sigma_{\log}^{C^\infty}$$

Thus, the (exponential) helicoid is C^∞ -diffeomorphic to the Logarithmic Riemann surface, when the latter is equipped with the induced smooth sub-atlas $\mathcal{A}_{\Sigma_{\log}}^{C^\infty}$ of the holomorphic $\mathcal{A}_{\Sigma_{\log}}^{\mathcal{O}}$ coming from \mathbb{R}^4 .

The diffeomorphism Ξ will be constructed in the proof. As a consequence, 2.8 implies that we can equip the helicoid surface $\text{Exp}_a(\mathbb{C})$ with a holomorphic structure via the bijective correspondence of it with Σ_{\log} , that is, if (U, ϕ) is a holomorphic chart of $\mathcal{A}_{\Sigma_{\log}}^{\mathcal{O}}$, then define an atlas of $\text{Exp}_a(\mathbb{C})$ as follows

$$\mathcal{A}_{\text{Exp}_a(\mathbb{C})}^{\mathcal{O}} := \{(\Xi^{-1}(U), \Xi^{-1} \circ \phi) \mid (U, \phi) \in \mathcal{A}_{\Sigma_{\log}}^{\mathcal{O}}\}$$

Of course, this holomorphic atlas of the helicoid does not coincide with its induced smooth atlas coming from the ambient 3- space.

Proof. $\text{Exp}_a(\mathbb{C})$ is given as the following subset of \mathbb{R}^3 :

$$\{(e^u \cos v, e^u \sin v, av) \mid u, v \in \mathbb{R}\}$$

and Σ_{\log} as the subset of \mathbb{R}^4

$$\{(u, v, e^u \cos v, e^u \sin v) \mid u, v \in \mathbb{R}\}$$

Define a map Ξ as

$$\Xi : (e^u \cos v, e^u \sin v, av) \longmapsto (u, v, e^u \cos v, e^u \sin v)$$

Simple algebraic manipulations show that this map is a bijection with inverse the projection onto \mathbb{R}^3 composed with a rigid motion

$$\Omega \equiv \Xi^{-1} : (u, v, e^u \cos v, e^u \sin v) \longmapsto (e^u \cos v, e^u \sin v, av)$$

both maps can be easily seen to have smooth components and this completes the proof. \square

3. Multi-dimensional Generalizations.

In this section we present the reasonable high-dimensional analogues of the results presented in the previous *Sections*. Notwithstanding, the new context suggests that the results will no longer stand in the region of Riemann Surfaces, but will be of a more general differential-geometric nature.

The following definitions arise naturally from the respective 1.1, 1.3 of the *Section 1*.

3.1 Definition. (Multi-Exponential field) The (*exponential*) *multi-helicoid* is (in complex coordinates) the map $(a_1, \dots, a_m > 0$ parameters):

$$\text{Exp}_{a_1, \dots, a_m} : \mathbb{C}^m \longrightarrow (\mathbb{C} \setminus \{0\})^m \times \mathbb{R}^m :$$

defined as

$$(z_1, \dots, z_m) \longmapsto (\exp_{\mathbb{C}}(z_1), \dots, \exp_{\mathbb{C}}(z_m); a_1 \text{Im}(z_1), \dots, a_m \text{Im}(z_m))$$

that is, (up to a natural identification $\mathbb{C}^m \times \mathbb{R}^m \cong (\mathbb{C} \times \mathbb{R})^m$)

$$\text{Exp}_{a_1, \dots, a_m} := \prod_{1 \leq k \leq m} \text{Exp}_{a_k}$$

This is a vector field by 1.4 (used component-wise), defined on the (flat) trivially embedded submanifold \mathbb{C}^m of \mathbb{R}^{3m} and valued in (the tangent bundle of) the ambient \mathbb{R}^{3m} by 1.2.

Using 1.5, $\text{Exp}_{a_1, \dots, a_m}(\mathbb{C}^m)$ is a globally coordinated smooth submanifold of (real) dimension $\dim_{\mathbb{R}}(\text{Exp}_{a_1, \dots, a_m}(\mathbb{C}^m)) = 2m$. In fact, a **multi-helicoid**, in complete analogy with 1.1.

The proofs of the following results go without saying using the respective results of the previous sections, so we will not bother showing anything explicitly. A component-wise argument suffices in all cases, due to the product form of the definition 3.1.

If we set $a_1 = \dots = a_m = 1/n$, we obtain a multi-sequence of vector fields and in turn of multi-helicoid submanifolds of \mathbb{R}^{3m} :

$$\text{Exp}_{n,\dots,n}(z_1, \dots, z_m) := (\exp_{\mathbb{C}}(z_1), \dots, \exp_{\mathbb{C}}(z_m); \frac{1}{n}Im(z_1), \dots, \frac{1}{n}Im(z_1))$$

We are now in position to present the analogous result of Theorem 2.2.

3.2 Theorem. (*Convergence to Injective multi-Exponential*)

The sequence of exponential vector fields $(\text{Exp}_{n,\dots,n})_{n \in \mathbb{N}}$ converges point-wise to a C^∞ - diffeomorphism $\prod_m \exp_{\mathbb{C}} : \mathbb{C}^m \longrightarrow (\mathbb{C} \setminus \{0\})^m$:

$$\begin{array}{ccc}
 \begin{array}{c} \vdots \\ \searrow \\ \mathbb{C}^m \end{array} & \xrightarrow[\cong]{\text{Exp}_{n-1,\dots,n-1}} & \begin{array}{c} \text{Exp}_{n-1,\dots,n-1}(\mathbb{C}^m) \\ \swarrow \cong \\ \text{Exp}_{n,\dots,n}(\mathbb{C}^m) \end{array} \\
 \downarrow \Phi \cong & \searrow \cong & \downarrow \Psi \\
 \mathbb{C}^m & \xrightarrow[\cong]{\text{Exp}_{n,\dots,n}} & \text{Exp}_{n,\dots,n}(\mathbb{C}^m) \\
 \downarrow \Phi \cong & \searrow \Psi & \downarrow \Psi \\
 \lim_{\rightarrow} \mathbb{C}^m \cong \mathbb{C}^m & \xrightarrow[\cong]{\lim_{\rightarrow} \text{Exp}_{n,\dots,n}} & \lim_{\rightarrow} \text{Exp}_{n,\dots,n}(\mathbb{C}^m) \\
 \downarrow \prod_m \exp_{\mathbb{C}} & & \downarrow \cong \\
 & \xrightarrow{\prod_m \exp_{\mathbb{C}}} & \prod_m \exp_{\mathbb{C}}(\mathbb{C}) = (\mathbb{C} \setminus \{0\})^m
 \end{array}$$

The convergence is uniform on the bounded multi-strips $B^m \equiv \{z_k \in \mathbb{C} : |Im(z_k)| < M_k / M_k > 0, 1 \leq k \leq m\} \subseteq \mathbb{C}^m$.

$$\text{Exp}_{n,\dots,n}|_{B^m} \xrightarrow[U]{n \rightarrow \infty} \prod_m \exp_{\mathbb{C}} \times \{0\}|_B \equiv \prod_m \exp_{\mathbb{C}}|_B$$

3.3 Remark. The methods we expounded for these specific construction raise the question if and how they may be extended and applied to other surfaces, in the framework of a general approach that would, at least, be in position to give back the already known facts concerning the classical Riemann surfaces, e.g. those of $\log_{\mathbb{C}}(z)$ and $\sqrt[n]{z}$ which, the latter, more or less constituted the beginnings of the subject of the study of holomorphic 1-manifolds.

Our problem is focused in the quest of an appropriate sequence of surfaces in \mathbb{R}^3 (if any), whereon the complex function in question (properly extended as a vector field) will become single valued, and the covering surface will be obtained in a uniform limit.

The above attractive concept might be of significant importance in the finite-sheeted Riemann surfaces of algebraic functions satisfying the general formula $a_n(z)[f(z)]^n + a_{n-1}(z)[f(z)]^{n-1} + \cdots + a_0(z) = 0$.

In a nutshell, recovering via differential geometry pure analytic information concerning complex functions through their Riemann surfaces would give a measure of the amount that (the latter) in fact depend on the holomorphic structure of the plane and, more generally, of every complex manifold modeled upon it.

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